SUFFICIENT CONDITIONS FOR OPTIMIZATION

(K DOSTATOCHNYM UCLOVIIAM OPTIMAL'NOSTI)

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We will analyse the problem described in article [1] and dealing with the determination of an optimal trajectory. In our work we will observe the same terminology and the same system of notations as those used in [1]. In Section 4 of [1] those sufficient conditions (Theorem 4.1) are indicated which are such that a trajectory $x(x_0,\ t,\ \eta_0)$ in the system of equations

$$\frac{dx}{dt} = f(x,\ t) + q\eta(t) \tag{1}$$

is locally optimal (in the sense of definition 4.1 in [1]) in respect of the admissible governing functions $\eta(t)$, constrained by the condition

$$|\eta(t)| \leqslant 1 \tag{2}$$

Apart from conditions 1 to 4 (which are analogous to conditions given by the principle of a maximum [2]), Theorem 4.1 of [1] contains some other limitations (shown by expression 5) on the second derivatives of $\partial^2 f_i/\partial x_j \partial x_k$. It is the aim of this note to prove that, without these additional limitations, conditions 1 to 4 of [1] fail to produce a locally optimal trajectory $x(x_0,\ t,\ \eta^0)$, and that even these conditions are insufficient for the trajectory to become optimal in respect of small variations of function $\eta^0(t)$ constrained by condition (2).

Let us now analyse the system of equations

$$\frac{d\xi}{dt} = \zeta + \alpha_1(\xi,\ \zeta), \qquad \frac{d\zeta}{dt} = -\xi + \alpha_2(\xi,\ \zeta) + \eta \tag{3}$$

where α_1 and α_2 are some nonlinear, sufficiently smooth functions which will be presented below. Together with this nonlinear system (3) we will analyse an auxiliary linear system of equations

$$\frac{d\xi}{dt} = \zeta, \qquad \frac{d\zeta}{dt} = -\xi + \eta \tag{4}$$

It is known [3] that the optimal trajectory of the linear system (4), which connects point $\xi = -3$, $\zeta = 0$ with point $\xi = 0$, $\zeta = 0$ (under the limitations imposed by (2)), is as shown on Fig. 1. In Fig. 1 line *AB* represents a circular arc with its center at point (1, 0), *BC* represents a circular arc with its center at point (−1, 0), and *CO* represents a circular arc with its center at point (1, 0). The corresponding optimal governing function $\eta^0(t)$ has the form

$$\eta^0(t) = -\operatorname{sign}[\sin(t - \psi_0)]$$

Let us designate by $\rho(\xi, \zeta)$ the distance from point (ξ, ζ) to point $(\xi = -1, \zeta = 0)$, by $\phi(\xi, \zeta)$ the angle between axis $\zeta = 0$ and a ray connecting point $(\xi = -1, \zeta = 0)$ with point (ξ, ζ), by $\psi(\xi, \zeta)$ the angle between axis $\zeta = 0$ and a ray from point $(\xi = 1, \zeta = 0)$ to point (ξ, ζ).

We can now determine the functions $a_1(\xi, \zeta)$, $a_2(\xi, \zeta)$ in the following manner:

$$\alpha_1 \equiv 0, \ \alpha_2 \equiv 0$$

everywhere outside of zone *G* enclosed by rays $\phi_1 = \pi/4$, $\phi_2 = 3\pi/4$ (zone *G* is crosshatched in Fig. 1); within zone *G* functions a_1 and a_2 are determined by formulas

$$\begin{aligned}\alpha_1(\xi, \zeta) &= \omega\zeta\rho^4(\rho - \rho_0)^2 \exp[(\varphi - \varphi_1)(\varphi - \varphi_2)]^{-1}\\ \alpha_2(\xi, \zeta) &= -\omega(\xi + 1)(\rho - \rho_0)^2\rho^4 \exp[(\varphi - \varphi_1)(\varphi - \varphi_2)]^{-1}\end{aligned} \qquad (5)$$

Here $\omega = \text{const} > 0$ and the value of ρ_0 is indicated on Fig. 1.

Obviously at $\eta = -1$ the trajectories of system (3) and the trajectories of system (4) appear within zone *G* as circular arcs with their centers at a point with coordinates $\xi = -1$, $\zeta = 0$.

It should be noted that in the vicinity of curve *ABCO* the functions a_1 and a_2 *are of the second order of smallness, and that only the first* derivatives of $\partial f_i/\partial x_j$ enter into conditions 1 to 4 of Theorem 4.1 of [1]. For these reasons the trajectory *ABCO*, which is optimal for the linear system (3), satisfies (as can be easily verified) conditions 1 to 4. (We should also note that this trajectory satisfies conditions of the principle of a maximum [2]).

We can now verify by direct calculations that it is possible, at sufficiently large values of $\omega > 0$, to choose an admissible function $\eta(t)$ in *such a way that it would satisfy conditions* (2) and which, at the same time, would correspond to the trajectory $\xi(\xi_0, \zeta_0, t, \eta)$, $\zeta(\xi_0, \zeta_0, t, \eta)$ of system (3), which connects point $(\xi_0 = -3, \zeta_0 = 0)$ and point $(\xi = 0, \zeta = 0)$; this trajectory would also have a smaller time length than the trajectory *ABCO*.

Moreover, the trajectory $\xi(\xi_0, \zeta_0, t, \eta)$, $\zeta(\xi_0, \zeta_0, t, \eta)$ may pass as near as is desired to the curve *ABCO*, while the variations of $\delta\eta$ of function $\eta^0(t)$ (that is, the magnitudes of $\delta\eta = \eta(t) - \eta^0(t)$) may be as small

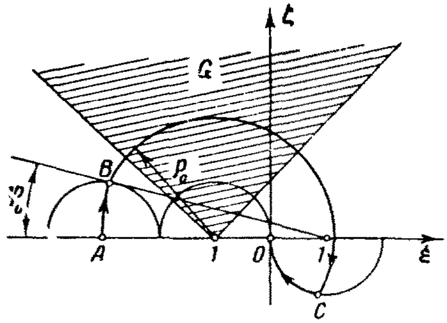

Fig. 1.

as is desired. We will not include here the details of all such calculations, but we will establish the correctness of our assertions by means of simple graphic representations.

First, we will assume that function $\eta(t)$ is replaced in the right side of system (3) by function

$$\eta_1(t) = \eta^0(t) + \varepsilon\mu\delta(t - \psi_0) + \varepsilon\gamma\delta(t - \psi_0 - \pi)$$

where $\delta(t)$ represents the δ-function, μ, γ are some sufficiently small positive constants, and $\epsilon > 0$ is some arbitrarily small constant, given *a priori*.

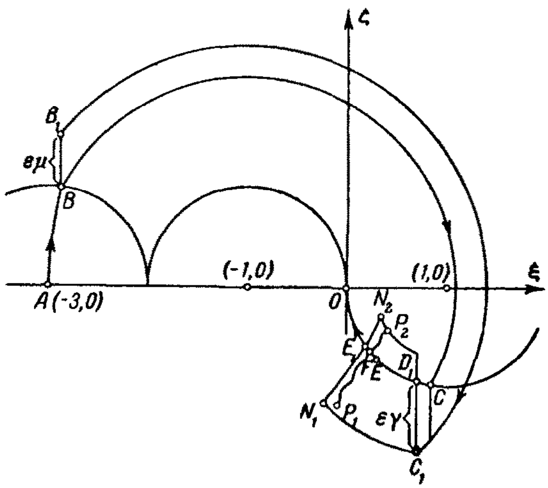

Fig. 2.

Outside zone G we have $d\phi/dt = -1$ at $t \neq \psi_0$, $t \neq \psi_0 + \pi$; within zone

G in the vicinity of curve $ABCO$ we have

$$\frac{d\varphi}{dt} \leqslant -1$$

$$\frac{d\varphi}{dt} < -1 - \beta\omega(\rho - \rho_0)^2$$

$$\left(\frac{1}{3}\pi < \varphi < \frac{1}{2}\pi\right) \qquad (6)$$

where β is a positive constant. For the above reasons the trajectory of system (3), which corresponds to such a function $\eta_1(t)$ and which passes through point $\xi_0 = -3$, $\zeta_0 = 0$ at $t = 0$ (when μ and γ ($\mu \approx \gamma$) are properly chosen) will be of the form $AB_1C_1D_1O$ shown on Fig. 2.

This trajectory will reach some point E_1 (Fig. 2) at an instant $t = T$ where T_1 is smaller than the period of time $t = T$ needed by a point moving along the trajectory $ABCO$ to reach point E_1. From (6) we obtain the expression

$$T_1 - T \leqslant -\omega\nu_1\mu^2\varepsilon^2 \qquad (\nu_1 > 0 - \text{const}) \tag{7}$$

If we designate by E that point which is reached by the trajectory $ABCO$ at $t = T_1$, we will also obtain the inequality

$$\xi_E - \xi_{E_1} \geqslant \nu\omega\mu^2\varepsilon^2 \qquad (\nu > 0 - \text{const}) \tag{8}$$

For the time being we will consider μ as a fixed quantity and we will allow γ to vary within the limits of

$$0 < \gamma < 2\mu \tag{9}$$

In this case the termini of the trajectories of system (3), which at $t = 0$ come out of point $\xi_0 = -3$, $\zeta_0 = 0$, and which correspond to the instant $t = T_1$, will lie on the segment N_1N_2 of some curve (Fig. 2).

Let us now examine functions $\lambda_1(t)$, $\lambda_2(t)$ defined by formulas

$$\lambda_1(t) = \begin{cases} 0 & \text{for } t \leqslant \psi_0 \\ \frac{1}{\mu} & \text{for } \psi_0 < t \leqslant \psi_0 - \mu \\ 0 & \text{for } \psi_0 + \mu < t \end{cases}$$

$$\lambda_2(t) = \begin{cases} 0 & \text{for } t \leqslant \psi_0 + \pi - \mu \\ \frac{1}{\mu} & \text{for } \psi_0 + \pi - \mu < t \leqslant \psi_0 + \pi \\ 0 & \text{for } \psi_0 + \pi < t \end{cases} \tag{10}$$

We will construct the trajectories of system (3) starting at point $\xi_0 = -3$, $\zeta_0 = 0$ at $t = 0$ and corresponding to the governing function

$$\eta(t) = \eta^0(t) + \varepsilon\mu\lambda_1(t) + \varepsilon\gamma\lambda_2(t)$$

Obviously, function $\eta(t)$ will represent the admissible governing functions satisfying conditions (2) at any $\epsilon \ll 1$. The termini of the trajectories $\xi(\xi_0, \zeta_0, t, \eta)$, $\zeta(\xi_0, \zeta_0, t, \eta)$, which correspond to moment the $t = T_1$ for $0 < \gamma < 2$, and which at all the fixed values of $\mu > 0$, $\epsilon > 0$, will lie on some continuous curve P_1P_2. The distance between the points on this curve and the points on segment N_1N_2 will be of the second order of smallness in their μ and will satisfy the inequality (for small values)

$$\begin{aligned} |\xi(\xi_0, \zeta_0, t, \eta) - \xi(\xi_0, \zeta_0, t, \eta_1)| &\leqslant \Delta\varepsilon^2\mu^2 \\ |\zeta(\xi_0, \zeta_0, t, \eta) - \zeta(\xi_0, \zeta_0, t, \eta_1)| &\leqslant \Delta\varepsilon^2\mu^2 \end{aligned} \qquad (\Delta = \text{const}) \qquad (11)$$

where the constant Λ is independent of ω.

We can now conclude from the inequality (8) that when the chosen quantity ω is sufficiently large and when all the values of $\mu > 0$, $\epsilon > 0$ are sufficiently small, the curve P_1P_2 will intersect arc CO at point F lying farther to the left than point E. This fact indicates that the corresponding trajectory $ABFO$ of system (3) reaches point $\xi = 0$, $\zeta = 0$ in a shorter period of time then does a point moving along the trajectory $\xi(\xi_0, \zeta_0, t, \eta^0)$, $\zeta(\xi_0, \zeta_0, t, \eta^0)$ which is controlled by function $\eta_0(t)$.

Our assertion is proved.

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